Série 4b Solutions

Question 4b.1 – Maximal in plane shearing stress

We consider a material for which it is known that failure is due to shear stress, and its maximal shear stress before failure (shear yield stress, τ_{yield}) has been measured to be 75 MPa. A mechanical piece made of this material is submitted to the plane stresses shown in Figure 4b.1.

Determine the values of σ_y for which material failure due to maximum shear stress is observed.

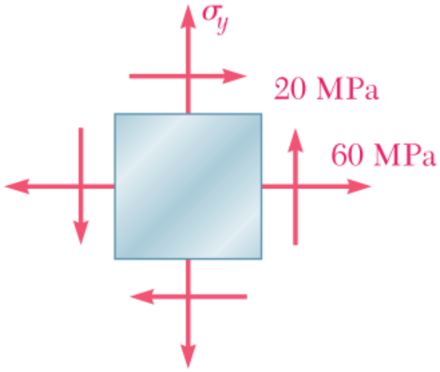


Figure 4b.1 – State of stress on a 2D element with unknown σ_y

Solution 4b.1

Objectives – what is asked?

The normal stress in the y direction at material failure due to shear stress

What is given?

A state of stress, where the normal stresses in x (σ_x = 60 MPa) and the shearing stress (τ_{xy} = 20 MPa) are known.

The maximum shearing stress before failure τ_{yield}

Principles and formulas

Maximum in-plane shearing stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1}$$

We know that failure happens due to shearing, and the shearing stress related to failure is known as well, so we set:

$$\tau_{max} = \tau_{yield} \tag{2}$$

From this, we solve for σ_y:

$$\left(\frac{\sigma_x - \sigma_y}{2}\right) = \pm\sqrt{\tau_{yield}^2 - \tau_{xy}^2} \tag{3}$$

$$\sigma_y = \sigma_x \pm 2\sqrt{\tau_{yield}^2 - \tau_{xy}^2} \tag{4}$$

Calculation

$$\sigma_{y,max} = 60 \pm 2\sqrt{75^2 - 20^2} = \begin{cases} -84.57\ MPa \\ 204.57\ MPa \end{cases} \tag{5}$$

Yield due to shearing stress will happen for values of σ_y lower than $-84.57\ MPa$ and larger than $204.57\ MPa$.

Question 4b.2 – Axial load on glued joint

Two members of uniform cross section 50 x 80 mm are glued together along plane 'a-a' that forms an angle of 25° with the horizontal x-axis (Figure 4b.2). We consider that the material of the beam is tough and does not break, and expect failure in the glued joint instead. We know that the yield stresses for the glued joint are different for the normal stress on the joint ($\sigma_{\perp,yield}$) and for the shearing stress parallel the joint ($\tau_{\parallel,yield}$). They are respectively $\sigma_{\perp,yield}$ = 800 kPa and $\tau_{\parallel,yield}$ = 600 kPa.

Determine the largest centric load F that can be applied before failure in the glued joint (we consider it a 2D problem).

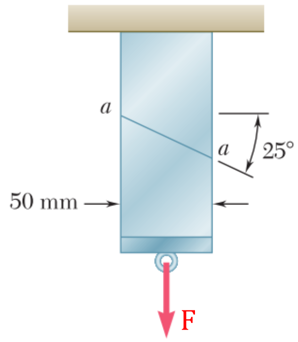


Figure 4b.2 – Vertical load on a glued joint

Solution 4b.2

Objectives – what is asked?

Determine the maximal load F that can be applied before the stresses in the joint surpasses the allowable stresses

What is given?

The dimensions of cross-section: 50 x 80 mm

The angle of the joint with respect to the horizontal: 25° clockwise

The allowable stresses for the joint (normal stress $\sigma_{\perp,yield}$ = 800 kPa and shear stress $\tau_{\parallel,yield}$ = 600 kPa)

Principles and formulas

Consider the formulas for the rotation of stresses on an element:

$$\begin{aligned} \sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos(2\theta) + \tau_{xy}\, sin(2\theta) \\ \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos(2\theta) - \tau_{xy}\, sin(2\theta) \\ \tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) \end{aligned} \quad (1)$$

The normal stress on a section A with an applied load F is simply given by:

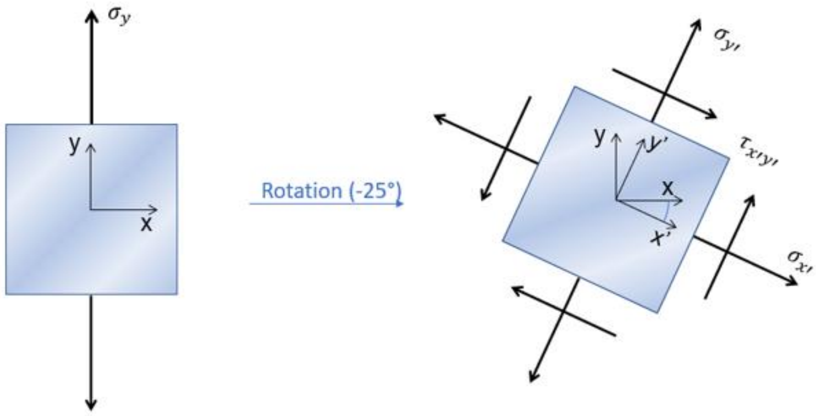
$$\sigma = \frac{F}{A} \quad (2)$$

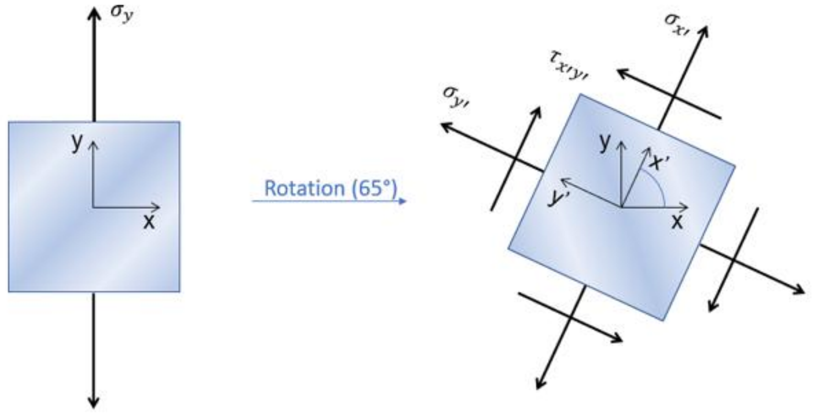
Calculations

The first step is to determine the state of stress of an element oriented in the x-y axes. Only a normal tensile force in y is applied to the system, therefore the state of stress can be written as follows:

$$\begin{cases} \sigma_x = 0 \\ \sigma_y = F/A \\ \tau_{xy} = 0 \end{cases} \quad (3)$$

The next step is to determine the angle with which to rotate this element to obtain the normal stresses and the shearing stresses in the joint. This can be looked at two ways. Consider the drawing below:





If the element is rotated -25°, the normal stress on the joint will be $\sigma_{y'}$ and if the element is rotated 65° the normal stress on the joint will be $\sigma_{x'}$. The shearing stress on the joint will be $\tau_{x'y'}$ in both cases, only the sign will change. Let's consider $\theta = -25°$ for the calculations that follow.

$$\sigma_{\perp,yield} = \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos(2\theta) - \tau_{xy}\, sin(2\theta) \tag{4}$$

Using the stresses from eq (3) we can simplify this equation.

$$\sigma_{\perp,yield} = \frac{F}{2A}(1 + \cos(2\theta)) \tag{5}$$

Therefore, the maximum load F before failure due to normal stress on the joint is:

$$F_{max,\sigma} = \frac{2\sigma A}{1 + \cos{(2\theta)}} \tag{6}$$

The same calculations are done for the shear stress (from eq (1)):

$$\tau_{\parallel,yield} = \tau_{x'y'} = -\frac{-F}{2A}\sin(2\theta) \tag{7}$$

$$F_{max,\tau} = \frac{2\tau A}{\sin{(2\theta)}} \tag{8}$$

Numerical application

For the normal stress:

$$F_{max,\sigma} = \frac{2 * 800 * 10^3 * (50 * 10^{-3} * 80 * 10^{-3})}{1 + \cos(2 * -25°)} = 3896\ N \tag{9}$$

For the shearing stress:

$$F_{max,\tau} = \frac{2 * 600 * 10^3 * (50 * 10^{-3} * 80 * 10^{-3})}{\sin(2 * -25°)} = -6266\ N \tag{10}$$

Note: a force in compression is found to cause a shear in that direction, and a tensile load would create a shear in the opposite direction but with the same magnitude. In any case, a load of smaller magnitude is found to cause failure due to normal stress in the joint (eq (9)). In consequence, the maximal allowed load on this joint is F=3896 N.

Question 4b.3 – Tilted strain gauge

An Aluminum plate with Young modulus $E = 72$ GPa and Poisson's ratio $\nu = 0.33$ is loaded in biaxial stress by normal stresses σ_x and σ_y (see Figure 4b.3 below). A strain Gauge is bonded to the plate at an angle of 21° . The stress $\sigma_x = 86.4$ MPa. The gauge factor of the strain gauge is $GF = 50$. The relative electrical resistance variation observed is $47.3 \cdot 10^{-3}$.

$$GF = \frac{1}{\varepsilon_{along}} \frac{\Delta R}{R}$$

(a) What is the maximum in-plane shear-stress τ_{max}?
(b) What are the axial strain parameters along x,y and z?

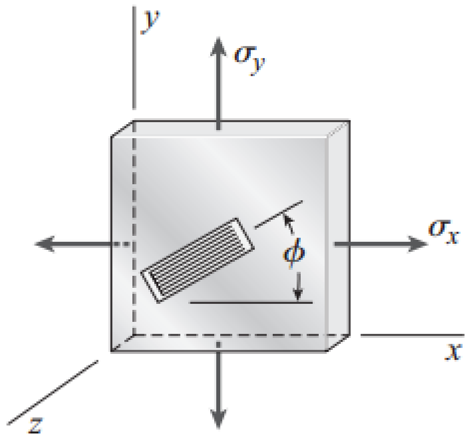


Figure 4b.3 – Strain gauge on a 3D element

Solution 4b.3

Obectives – what is asked?

(a) The maximum in-plane shear stress
(b) The axial strain parameters along x, y, z

What is given?

Young's modulus $E = 72$ GPa
Poisson's ratio $\nu = 0.33$
Resistance relative variation $\frac{\Delta R}{R} = 47.3 \cdot 10^{-3}$
Gauge Factor $GF = 50$
Stress submitted to along x-direction $\sigma_x = 86.4$ MPa
Angle of strain gauge $\phi = 21°$

Principles and formulas

We give the three dimensions Hooke's law for the strain in the x direction

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \tag{1}$$

Where, E is the Young's modulus of the material, ν is the Poisson's ratio. ϵ_x is the axial strain in the x-direction, and σ_x is the normal stress parallel to the x-axis. The Gauge's factor definition is given by :

$$GF = \frac{1}{\varepsilon}\frac{\Delta R}{R} \tag{2}$$

Where ε is the strain acting in the direction of the strain gauge. $\frac{\Delta R}{R}$ is the relative resistance variation.

Calculations

The strain in the direction of the strain Gauge is given by

$$\varepsilon_{x'} = \frac{1}{GF}\frac{\Delta R}{R} \tag{3}$$

where:

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\phi) + \frac{\gamma_{xy}}{2}\sin(2\phi) \tag{4}$$

To determine ε_x and ε_y, we use generalized Hooke's law. $\sigma_Z = 0$, so we do not consider its contribution.

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{5}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{6}$$

There is no shear strain along the (xy) plane (biaxial stress along x and y), meaning $\gamma_{xy} = 0$. Then using Equation (4),

$$\varepsilon_{x'} = \frac{1}{2E}[\sigma_x(1 - \nu + (1 + \nu)\cos(2\phi)) + \sigma_y(1 - \nu - (1 + \nu)\cos(2\phi))] \tag{7}$$

Then, we manipulate this equation in order to extract σ_y

$$\sigma_y = -\frac{\sigma_x(1-\nu+(1+\nu)\cos(2\phi))}{(1-\nu-(1+\nu)\cos(2\phi))} + 2E\frac{\varepsilon_{x'}}{(1-\nu-(1+\nu)\cos(2\phi))} \tag{8}$$

The maximum in-plane shear stress is extracted from the latter formula:

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2} \tag{9}$$

Every term of this last formula being known, we compute them in order to obtain the latter equation.

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2} = \frac{\sigma_x}{2} + \frac{1}{2}\left(\frac{\sigma_x(1-\nu+(1+\nu)\cos(2\phi))}{(1-\nu-(1+\nu)\cos(2\phi))} - 2E * \frac{\frac{1}{GF}\frac{\Delta \mathrm{R}}{\mathrm{R}}}{(1-\nu-(1+\nu)\cos(2\phi))}\right) \tag{10}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{11}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{12}$$

$$\varepsilon_z = \frac{1}{E}(-\nu(\sigma_x + \sigma_y)) \tag{13}$$

Numerical application

(a)

$$\tau_{max} = 32.1 \text{ MPa} \tag{14}$$

(b)

$$\varepsilon_x = 1.10 \cdot 10^{-3} \tag{15}$$

$$\varepsilon_y = -88 \cdot 10^{-6} \tag{16}$$

$$\varepsilon_z = -497 \cdot 10^{-6} \tag{17}$$

Question 4b.4 – Temperature change in beam

Part 1:

A beam, clamped of both ends, with a square cross section, shown in Figure 4b.4 (Part 1), has a Young's Modulus $E = 200$ GPa, coefficient of thermal expansion $\alpha = 10^{-6}\ \mathrm{K}^{-1}$, thickness $h_0 = 1$ cm and length $L = 1$ m. The longitudinal yield stress of the beam is $|\sigma_{yield}| = 30$ MPa and the shear yield stress $|\tau_{yield}| = 10$ MPa. The beam undergoes a positive ΔT.

(a) What is the necessary ΔT to have material failure due to longitudinal stress?
(b) The failure will be due to compressive or tensile stress?
(c) For a given temperature change ΔT, what is the maximum shear stress and the angle of rotation θ at which this shear stress is found?
(d) What is the necessary ΔT to have material failure due to shear stress?

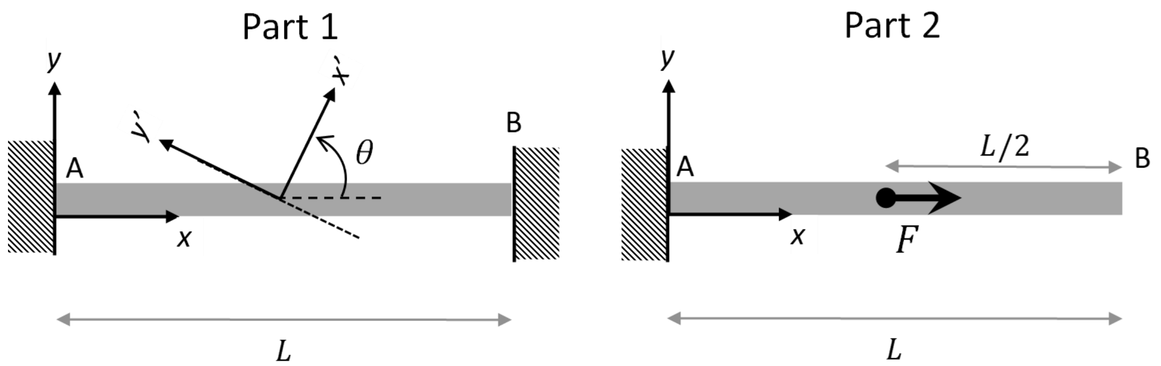


Figure 4b.4 – Part 1: Temperature change in a beam clamped on both ends – Part 2: Temperature change in a beam clamped on side with an applied load F

Part 2:

The beam is now only clamped on one end and undergoes also a force F applied in the middle point, as shown in Figure 4b.4 (Part 2).

(e) If $\Delta T = 50$ K, what is the maximum force F before failure?
(f) Is that failure due to longitudinal or shear stress?
(g) How much is the total elongation of the bar just before failure?

Solution 4b.4

Part (a)

In general, the total strain in the beam is equal to:

$$\varepsilon_{TOT} = 0 = \varepsilon_{th} + \varepsilon_{mech} \rightarrow \varepsilon_{mech} = -\varepsilon_{th} \tag{1}$$

Therefore, the longitudinal stress in the beam is equal to:

$$\sigma_{mech,long} = E\varepsilon_{mech} = -E\alpha\Delta T \tag{2}$$

To have failure, the longitudinal stress must be equal to or larger than the longitudinal yield stress. With this, we can find the positive temperature change to induce material failure in compression:

$$\sigma_{mech,long} = -\sigma_{yield} \rightarrow \Delta T = \frac{\sigma_{yield}}{E\alpha} \tag{3}$$

Final Answer:

$$\Delta T = \frac{30MPa}{200GPa * 10^{-6}} = \frac{30000}{200} = 150 \text{ K} \tag{4}$$

Part (b)

Compressive due to the positive change in temperature and positive coefficient of thermal expansion.

Part (c)

We know the formula for shear stress on a coordinate system at an angle θ to the original coordinate system**:**

$$\tau_{x',y'} = \frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \tag{5}$$

Because we have no shear stress nor stress in the y direction, this simplifies to:

$$\tau_{x\prime,y\prime} = \frac{\sigma_x}{2}\sin{(2\theta)} \tag{6}$$

Thus the maximum and minimum shear angles are:

$$\theta = \pm 45^{\circ} = \pm\frac{\pi}{4}. \tag{7}$$

The maximum and minimum shear stresses are equal to:

$$\tau_{Max,Min} = \pm\frac{\sigma_{mech,long}}{2} = \pm\frac{E\alpha\Delta T}{2} \tag{8}$$

Part (d)

In the same fashion as Part (a), we can find the change in temperature necessary to induce material failure due to shear stress. We use the equation found for the shear stress in Part C, equate it to the shear yield stress and solve for ΔT:

$$\tau_{Max,Min} = \tau_{yield} \rightarrow \Delta T = \frac{2\tau_{yield}}{E\alpha} \tag{9}$$

Final Answer:

$$\Delta T = \frac{20MPa}{200GPa * 10^{-6}} = \frac{20000}{200} = 100 \text{ K} \tag{10}$$

Part (e) and (f)

The temperature does not generate stress, so the stress is only generated by the force:

$$\sigma_F = \frac{F}{A} \rightarrow F_{Max} = \sigma_{yield} A \tag{11}$$

Failure can be due to either shear stress or longitudinal stress.

$$F_{long,Max} = \sigma_{yield} A = 30\ MPa \cdot 10^{-4} m = 3\ kN \tag{12}$$

$$F_{shear,Max} = \tau_{yield} 2A = 20\ MPa \cdot 10^{-4} m = 2\ kN \tag{13}$$

Failure will be due to shear stress.

Part (g)

The total elongation of the beam will be due to both the change in temperature and the applied force. Using the definition of strain and solving for the change in length:

$$\Delta L_{TOT} = \Delta L_F + \Delta L_{Th} \tag{14}$$

$$\Delta L_{Th} = \alpha \Delta T L \tag{15}$$

$$\Delta L_F = \frac{L}{2} \frac{F_{Max}}{EA} \tag{16}$$

$$\Delta L_{TOT} = \Delta L_F + \Delta L_{Th} = \frac{L}{2} \frac{F_{Max}}{EA} + \alpha \Delta T L = \frac{10^{-4}}{2} + 5 * 10^{-5} = 0.1\ \text{mm} \tag{17}$$

Question 4b.5 – Von Mises criterion and safety factor

The state of plane stress shown in Figure 4b.5 occurs in a machine component made of a steel with σ_{yield} = 315 MPa (the yield stress of that steel).

Using the Von Mises criterion, determine whether yield will occur when (a) τ_{xy} = 63 MPa, (b) τ_{xy} = 140 MPa. If yield does not occur, determine the corresponding Safety Factor (SF_{VM}).

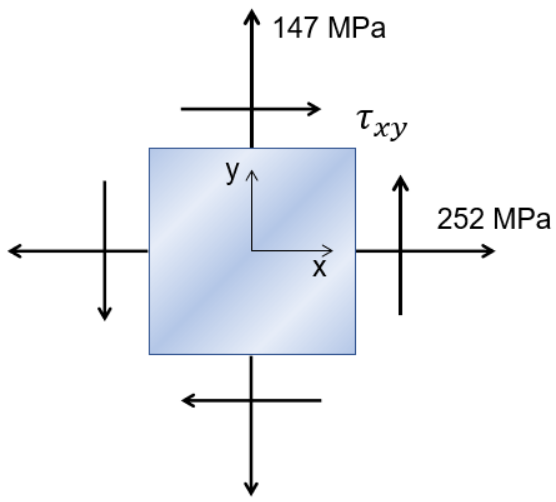


Figure 4b.5 – State of stress on a 2D element with varying τ_{xy}

Solution 4b.5

Objectives – what is asked?

If yield will occur in the material considering the Von Mises criterion (σ_{VM}), and if not, the corresponding safety factor SF_{VM}

What is given?

A state of stress, where the normal stresses in x (σ_x = 252 MPa, σ_y = 147 MPa) and the shearing stress (τ_{xy} = (a) 63 MPa, (b) 140 MPa) are known.

The yield stress σ_{yield}=315 MPa

Principles and formulas

The principal stresses:

$$\sigma_{max,min} = \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1)$$

The von Mises criterion in 3D:

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \quad (2)$$

The safety factor when using the Von Mises criterion:

$$SF_{VM} = \frac{\sigma_{yield}}{\sigma_{VM}} \quad (3)$$

Calculations

As a first step, the Von Mises criterion can be simplified to the two-dimensional case by setting σ_3=0 (the third principal stress will be 0 as we are working in the plane):

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2)^2 + (-\sigma_1)^2} \quad (4)$$

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 + \sigma_1^2} \quad (5)$$

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{2\sigma_1^2 - 2\sigma_1\sigma_2 + 2\sigma_2^2} \quad (6)$$

$$\sigma_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \quad (7)$$

Moreover, the principal stresses can be written as:

$$\sigma_{1,2} = \sigma_{ave} \pm R \tag{8}$$

With:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}, \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{9}$$

All variables are known, and we can determinate if yield occurs or not

Numerical application

$$\sigma_{ave} = \frac{252 + 147}{2} = 199.5\ MPa$$

(a) Principal stresses and Von Mises stress:

$$R = \sqrt{\left(\frac{252 - 147}{2}\right)^2 + 63^2} = 82.00\ \text{MPa} \tag{10}$$

$$\begin{aligned} \sigma_1 &= 199.5 + 82 = 281.5\ MPa \\ \sigma_2 &= 199.5 - 82 = 117.5\ MPa \end{aligned} \tag{11}$$

$$\sigma_{VM} = \sqrt{281.5^2 - 281.5 * 117.5 + 117.5^2} = 244.65\ \text{MPa} < 315\ \text{MPa} \tag{12}$$

Failure does not occur for this case, the safety factor is

$$SF_{VM} = \frac{315}{244.65} = 1.3 \tag{13}$$

Note: for most applications, this safety factor would still be considered too low. In general a safety factor >2 is desirable.

(b) Principal stresses and Von Mises stress:

$$R = \sqrt{\left(\frac{252 - 147}{2}\right)^2 + 140^2} = 149.52\ MPa \tag{14}$$

$$\begin{aligned} \sigma_1 &= 199.5 + 149.52 = 349.02\ MPa \\ \sigma_2 &= 199.5 - 149.52 = 49.98\ MPa \end{aligned} \tag{15}$$

$$\sigma_{VM} = \sqrt{349.02^2 - 349.02 * 49.98 + 49.98^2} = 326.91\ \text{MPa} > 315\ \text{MPa} \tag{16}$$

Yield occurs for this case.

OPTIONAL - Question 4b.6 – Coated AFM tip

The tips of atomic force microscopes (AFM) can be coated to increase their resistance to the loads applied on them. Consider a simplified representation of such a system, as seen in Figure 4b.6. The tip is made of silicon and coated with diamond. The dimensions of the tip and the thickness of the coating are indicated in Figure 4b.6 and have the following values: R = 0.5 µm, r = 0.2 µm, L = 3 µm, t = 50 nm. The values of the elastic modulus and yield stress for the silicon and the diamond are respectively E_{Si} = 140 GPa, $\sigma_{yield,Si}$ = 165 MPa and E_D = 1200 GPa, $\sigma_{yield,D}$ = 1.2 GPa. The AFM is used in contact mode, and the force applied on its tip is measured to be 50 nN. Consider that the internal forces in the silicon and the diamond are uniform and constant across the length of the tip.

Hint: consider that the thickness of the coating is much smaller than the radius of the tip at any point ($t \ll R, t \ll r$).

(a) Calculate the internal forces respectively in the silicon tip and in the coating part.
(a) What are the safety factors (SF) with this applied force for the tip and the coating, respectively?
(b) What would be the safety factor (SF) for the tip if the coating was not present?

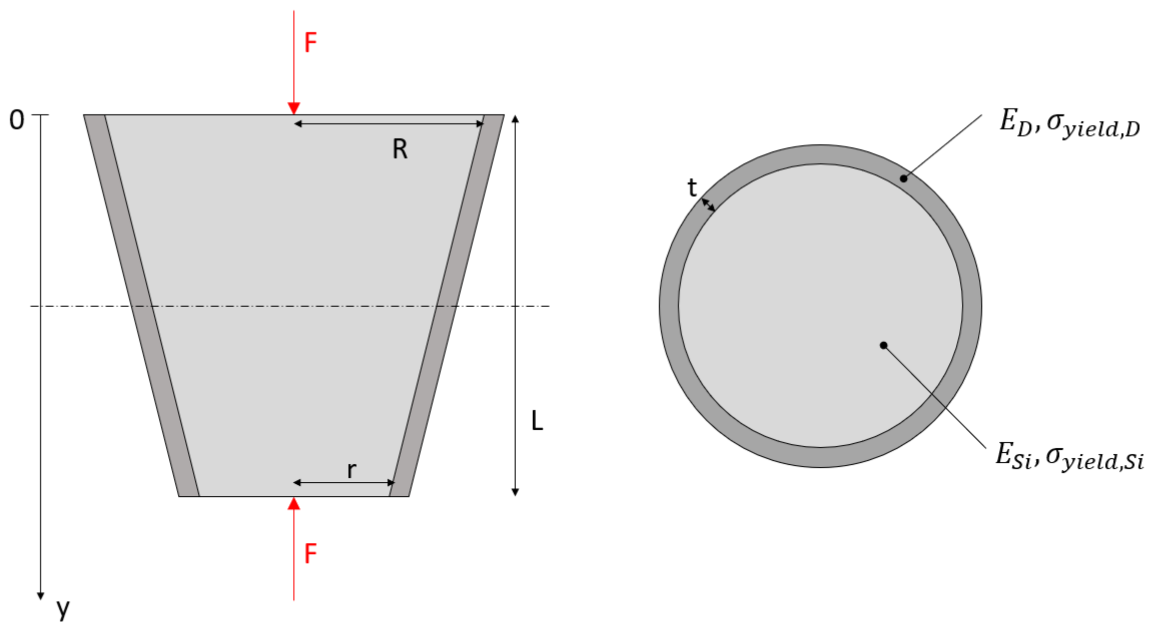


Figure 4b.6 – A simplified representation of a coated AFM tip

Solution 4b.6

Objectives – what is asked?

(a) The force F_{Si} in the silicon tip and the force F_D in the diamond coating
(b) The safety factor SF_{Si} for the tip and the safety factor SF_D for the coating

What is given?

The dimensions of the system are known:

- Smallest radius r
- Largest radius R
- Length of the tip L
- Coating thickness t

The mechanical properties of the materials are known as well:

- Young's modulus and yield stress of the silicon: E_{Si}, $\sigma_{yield,Si}$
- Young's modulus and yield stress of the diamond: E_D, $\sigma_{yield,D}$

Principles and formulas

The total deformation of a bar with varying cross section A, elastic modulus E and internal force N is given as:

$$\delta = \int_0^L \frac{N(x)}{A(x)E(x)} dx \tag{1}$$

In our case, the direction of interest is y and the internal force and the elastic modulus do not change over the length of the bar. The displacement has to be written for both the tip and the coating since their Young's modulus and cross-sections are different (i in the following equation can be 'Si' or 'D' for respectively the tip and the coating):

$$\delta_i = \int_0^L \frac{N_i}{A_i(y)E_i} dy \tag{2}$$

The internal forces in the tip and the coating are unknown and will be named N_{Si} and N_D respectively. Their sum is equal to the (compressive) force applied on the system, so we write:

$$N_{Si} + N_D = -F \tag{3}$$

Finally, we know that the displacement for the tip and the coating must be equal.

$$\delta_{Si} = \delta_D \tag{4}$$

Calculations

(a) Forces in the tip and coating

First, we write the displacement for the silicon:

$$\delta_{Si} = \int_0^L \frac{N_{Si}}{A_{Si}(y)E_{Si}} dy \tag{5}$$

The radius of the silicon tip as a function of the y coordinate can be written:

$$r_{Si}(y) = R - \frac{y}{L}(R - r) \tag{6}$$

And the area as a function of y:

$$A_{Si}(y) = \pi \left(R - \frac{y}{L}(R - r)\right)^2 \tag{7}$$

We rewrite the displacement and integrate:

$$\delta_{Si} = \int_0^L \frac{N_{Si}}{\pi\left(R - \frac{y}{L}(R-r)\right)^2 E_{Si}} dy \tag{8}$$

$$\delta_{Si} = \left[\frac{N_{Si}}{\pi E_{Si}\frac{(R-r)}{L}\left(R - \frac{y}{L}(R-r)\right)}\right]_0^L = \frac{N_{Si}L}{\pi E_{Si}(R-r)}\left(\frac{1}{r} - \frac{1}{R}\right) \tag{9}$$

We can follow the same steps for the coating. The cross-section area of the coating can be written (considering that t<<R, t<<r):

$$A_D(y) = 2\pi r_{Si}(y)t = 2\pi t(R - \frac{y}{L}(R-r)) \tag{10}$$

We can rewrite eq (2) for the coating and integrate it.

$$\delta_D = \int_0^L \frac{N_D}{2\pi t E_2\left(R - \frac{y}{L}(R-r)\right)} dy \tag{11}$$

$$\delta_D = \left[\frac{-N_D ln\left(R - \frac{y}{L}(R-r)\right)}{2\pi t E_D \frac{(R-r)}{L}}\right]_0^L = \frac{N_D L}{2\pi t E_D(R-r)}(\ln(R) - \ln(r)) \tag{12}$$

Using the fact that the displacements must be equal (eq (4)), we express N_{Si} as a function of N_D

$$N_{Si} = N_D \frac{E_{Si}(\ln(R) - \ln(r))}{2tE_D\left(\frac{1}{r} - \frac{1}{R}\right)} \tag{13}$$

Then, knowing that the forces must add up to -F (eq (3)), we can find the value for each force. We define the constant C as follows to simplify:

$$C = \frac{E_{Si}(\ln(R) - \ln(r))}{2tE_D\left(\frac{1}{r} - \frac{1}{R}\right)} \tag{14}$$

$$N_1 = \frac{-C}{1+C}F \tag{15}$$

$$N_2 = \frac{-1}{1+C}F \tag{16}$$

(b) Safety factors:
The maximal stresses in each component can be expressed as follows (we consider the stresses in the smallest areas (at y=L) as they will be maximal there):

$$\sigma_{ymax,Si} = \frac{N_{Si}}{A_{min,Si}} \qquad \sigma_{ymax,D} = \frac{N_D}{A_{min,D}} \tag{17}$$

There are stresses only in the y direction, no normal stress in x or shear stress is present. The safety factors are then trivial to calculate:

$$SF_{Si} = \left|\frac{\pi r^2 \sigma_{yield,Si}}{N_{Si}}\right| \qquad SF_D = \left|\frac{2\pi r t \sigma_{yield,D}}{N_D}\right| \tag{18}$$

(c) Safety factor for the uncoated tip:
The force F is in this case acting solely on the silicon, the reaction normal force in the silicon is N=-F. The safety factor is:

$$SF_{Si,nocoating} = \left|\frac{\pi r^2 \sigma_{yield,Si}}{-F}\right| \tag{19}$$

Numerical applications

(a) Forces in the tip and coating:

$$C = \frac{140 * 10^9 (\ln(0.5 * 10^{-6}) - \ln(0.2 * 10^{-6}))}{2 * 50 * 10^{-9} * 1200 * 10^9 \left(\frac{1}{0.2 * 10^{-6}} - \frac{1}{0.5 * 10^{-6}}\right)} = 0.3563 \quad (20)$$

$$N_{Si} = \frac{-0.3563}{1 + 0.3563} 50\ [nN] = -13\ nN \qquad N_D = \frac{-1}{1 + 0.3563} 50\ [nN] = -37\ nN \quad (21)$$

(b) Safety factors:

$$SF_{Si} = \left|\frac{\pi (0.2 * 10^{-6})^2 * 165 * 10^6}{-13 * 10^{-9}}\right| = 1600$$
$$SF_D = \left|\frac{2\pi * 0.2 * 10^{-6} * 50 * 10^{-9} * 1.2 * 10^9}{-37 * 10^{-9}}\right| = 2000 \quad (22)$$

(c) Safety factor without coating:

$$SF_{Si,nocoating} = \left|\frac{\pi (0.2 * 10^{-6})^2 * 165 * 10^6}{-50 * 10^{-9}}\right| = 400 \quad (23)$$

